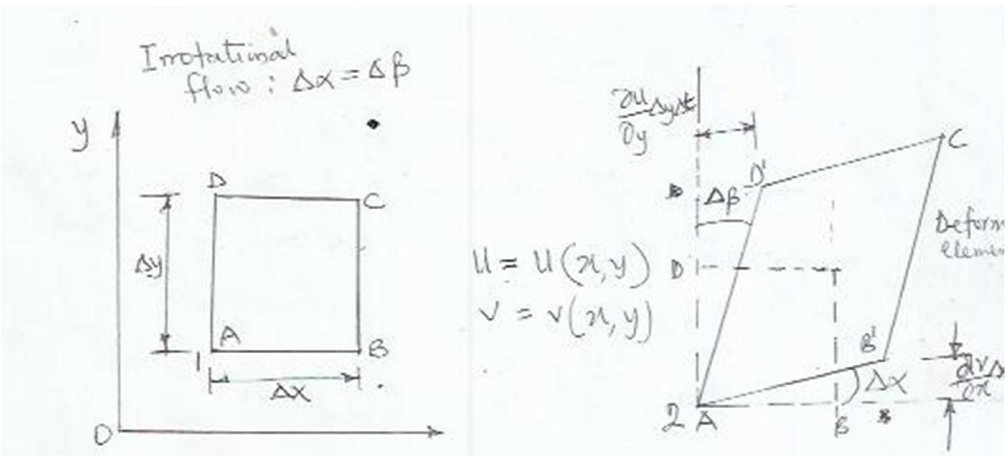
Irrotational flow, Vorticity and Circulation

Irrotational flow may be described as flow in which each element of the moving fluid suffers no net rotation from one instant to the next with respect to a given frame of reference. In flow along a curved path fluid elements will deform. If the axes of the element rotate equally towards or away from each other, then the flow will be irrotational. This means that as long as the algebraic average rotation is zero, the flow is irrotational.



Consider a rectangular fluid element of side dx and dy. Under the action of velocities acting on it let- it undergo deformation as shown above in a time Δt

Δx = Angular velocity of element AB = $\dfrac{dv}{dx}$

BB^1 = The Displacement $\dfrac{dv}{dx}\ \Delta x \Delta t$

ΔB = Angular Velocity of element AD = $\dfrac{du}{dy}$

ΔD^1 = The displacement $\dfrac{du}{dy}\ \Delta y \Delta t$

An element is shown moving from point 1 to point 2 along a curved path in the flow field. At 1 the unreformed elements is shown. As it moves to location2, the element is deformed. The angle of rotation of x axis is given by $\left(dv/dx\right). \Delta x.\Delta t$. The angle of rotation of y axis is given by $\left(du/dy\right). \Delta y.\Delta t$ It is assumed that $\Delta x.\Delta y$ i.e. the angle of rotation toward each other or away from each other should be equal.

The condition to be satisfied for irrotational flow is

$$\frac{dv}{dx} = \frac{du}{dy}. \; or \; \frac{dv}{dx} - \frac{du}{dy} = 0$$

In case there is rotation, then the rotation is given by (with respect to the Z axis in the case of two dimensional flow along x and y)

$$W_z = \tfrac{1}{2}\left({dv}/{dx} - {du}/{dy} \right)$$

and

$W_z = 0$ for irrotational flow.

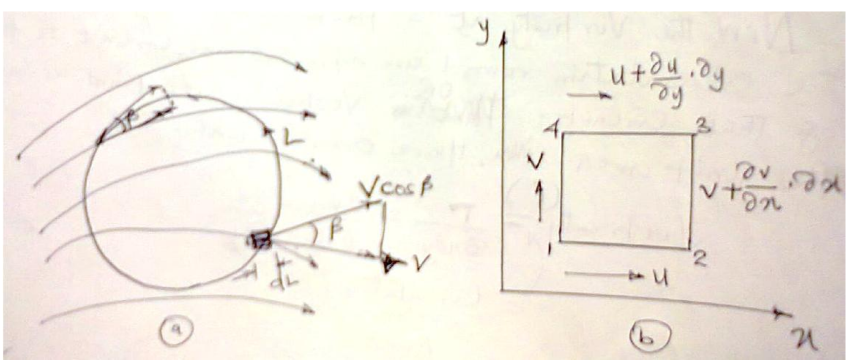
Concept of circulation and Vorticity

Considering a closed path in a flow field as shown below, circulation is defined as the line integral of velocity about this closed path. OR The circulation contained within a closed contour in a body of fluid is defined as the integral around the contour of the component of the velocity vector that is locally tangent to the contour. That is, the Circulation Γ is defined as

$$\Gamma = \oint u ds = \oint u \cos \beta dL$$

Where dl is the length on the closed curve, U is the velocity at the location and β is the angle between the velocity vector and the length DL.

The closed path may cut across several stream lines and at each point the direction of the velocity is obtained from the stream line, as its tangent at that point.



As an example of an elementary circuit arising from the subdivision of a larger one, we consider the elementary rectangle, $\partial x \times \partial y$ in size of the figure (b) above. The velocities along the sides have the directions and average values shown. Starting at the lower left-hand corner, we may add together the products of velocity and distance along each side, remembering that circulation is considered positive anticlockwise.

Consider the element 1234 in the figure (b) above, starting at 1 and proceeding counter clockwise,

$$\text{Circulation } \Gamma = u\partial x + \left(v + \frac{\partial v}{\partial x}\partial x\right)\partial y - \left(u + \frac{\partial u}{\partial y}\partial y\right)\partial x - v\partial y$$

$$= \frac{\partial v}{\partial x}\partial x\partial y - \frac{\partial u}{\partial y}\partial y\partial x$$

$$\text{OR} \qquad = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\partial x\partial y$$

Now, the vorticity at a point is defined as the ratio of the circulation round an infinitesimal circuit to the area of that circuit . OR Vorticity is defined as circulation per unit area. Here area $\partial x\partial y$, so

$$Vorticity(\xi) = \frac{d\Gamma}{dxdy} = \frac{dv}{dx} - \frac{du}{dy}$$

For irrotational flow vorticity and circulation are both zero. In polar coordinates.

$$Vorticity = \frac{dV_\theta}{dr} - \frac{1}{r}\frac{dV_r}{d_\theta} + \frac{V_\theta}{r}$$

Consider alternatively a small circular circuit of radius r ,

$$\Gamma = \oint uds = \oint \omega rrd\theta = r^2 \oint \omega d\theta = r^2\, \overline{\omega}\, 2\pi$$

Where $\overline{\omega}$ is the mean value of the angular velocity ω about the centre for all particles on the circle.

$$Vorticity(\xi) = \frac{d\Gamma}{dxdy} = \frac{r^2 \varpi 2\Pi}{\Pi r^2} = 2\varpi$$

That is, the vorticity at a point is twice the mean angular velocity of particles at that point. If the vorticity is zero at all points in a region of a flow (except certain special points, called singular points, where a velocity or the acceleration is theoretically zero or infinite) then the flow in that region is said to be irrotational. Flow in regions where the vorticity is other than zero is said to be irrotational.

Assignment

1. Show that the two-dimensional flow described (in metre per second units) by the equation $\psi = x + 2x^2 - 2y^2$ is irrotational. What is the velocity potential of the flow? If the density of the fluid is $1.12 kg/m^3$ and the piezometric pressure at the point (1, -2) is 4.8kPa. what is the piezometric pressure at the point (9,6)?

2. The stream function for a flow is given by $\psi = xy$. Is the flow irrotational ? Determine (i) u,v (ii) the vorticity and (iii) circulation.

3. Given that $u = x^2 - y^2$ and $v = -2xy$, determine the stream function and potential function for the flow.